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A gauge theoretical view of the charge concept in Einstein gravity *

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Abstract

We will discuss some analogies between internal gauge theories and gravity in order to better understand the charge concept in gravity. A dimensional analysis of gauge theories in general and a strict definition of *elementary*, *monopole*, and *topological* charges are applied to electromagnetism and to teleparallelism, a gauge theoretical formulation of Einstein gravity.

As a result we inevitably find that the gravitational coupling constant has dimension $\hbar/l^2$, the mass parameter of a particle dimension $\hbar/l$, and the Schwarzschild mass parameter dimension l (where l means *length*). These dimensions confirm the meaning of mass as elementary and as monopole charge of the translation group, respectively. In detail, we find that the Schwarzschild mass parameter is a *quasi-electric monopole charge of the time translation* whereas the NUT parameter is a *quasi-magnetic monopole charge of the time translation* as well as a topological charge. The Kerr parameter and the electric and magnetic charges are interpreted similarly. We conclude that each elementary charge of a Casimir operator of the gauge group is the source of a (quasi-electric) monopole charge of the respective Killing vector.

Keywords: gauge theory of gravity, Kaluza-Klein, charge, monopole, mass, Taub-NUT.

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1 Introduction

It was in the fifties when Yang & Mills [1] and Utiyama [2] formulated gauge theories of the group $SU(2)$ and of general semi-simple Lie groups, respectively. For the history see [3]. The great success of such theories has also influenced modern formulations of gravity. For a general formulation of gravity as a gauge theory we refer to [4]. A simpler introduction is [5]. There are many analogies between *internal* and *external* gauge theories, i.e. between Yang-Mills type theories and gauge theories of gravity (see table 1). The main difference is the fact that in external gauge theories one needs to solder the gauge to the base manifold, i.e. the gauge applies to spacetime and not to internal fibres. In standard formulations of gauge theories of gravity, this leads to the additional role of the coframe ϑ^α as a translational gauge potential.

Any gauge theory of gravity must include the translational gauge since otherwise gravity would not couple to the energy-momentum current. This is also natural since Minkowski space has an *affine* structure. In the following we will focus on the purely translational gauge theory, namely teleparallelism. With a specific lagrangian, this theory is equivalent to Einstein gravity. In section 2, a short dimensional analysis of general gauge theories leads to some interesting facts in the case of a translational gauge. The Kaluza-Klein formulation of electromagnetism makes a comparison with gravity very simple. In section 3 we will give the definitions of monopole, topological, and elementary charges and will discuss the meaning of mass as elementary charge. Finally, in section 4, we will analyze standard solutions of Einstein gravity of the Taub-NUT and Kerr-Newman type. We present the mass parameter as (quasi-electric) monopole charge and the NUT parameter as a (quasi-magnetic) monopole and topological charge.

2 Dimensional analysis of gauge theories

The essential fields involved in a gauge theory of a Lie group G (with algebra $\mathcal{G}$) are the connection A, the field strength F, the excitation H, the lagrangian $\mathcal{L}$, and the Noether current Σ. From a geometrical point of view, the connection is introduced as a $\mathcal{G}$-valued 1-form on the principle bundle or, locally, as a $\mathcal{G}$-valued 1-form on spacetime, i.e. $A \in \Lambda^1(M, \mathcal{G})$. It yields the covariant exterior derivative $D = d + A$.

By its very definition, the exterior differentiation operator d is dimensionless, $[d] = 1$. Hence we also require the connection to be dimensionless, $[A] = 1$. Now we need to give *exactly two* definitions in order to find all the remaining dimensions. First, we *choose to define* the dimension of a lagrangian $\mathcal{L}$ to be $[\mathcal{L}] = \hbar$. In the context of a classical gauge theory $\hbar$ is merely a name of a dimension as introduced here. However, thinking of Huygen's principle and the path integral method, one may also call $\hbar$ a *phase*/2π *unit*. And second, we define the basis elements λ_a of the algebra $\mathcal{G}$ to have the dimension $[\lambda_a] = g/\hbar$.

theory	gauge group	connection	field strength
general gauge theory	semi-simple Lie group G	$A \in \Lambda^1(M, \mathcal{G})$	$F = D\Gamma \in \Lambda^2(M, \mathcal{G})$
electrodynamics	$U(1)$	A	$F = dA$
(*non-physical*)	affine group	$\overset{\approx}{\Gamma}$	$\overset{\approx}{R}$
affine gauge theory	soldered affine group	$\Gamma_\alpha{}^\beta$, ϑ^α	$R_\alpha{}^\beta$, T^α
teleparallelism	soldered translations	ϑ^α	$T^\alpha = d\vartheta^a$

Table 1: Overview on gauge formalisms: Gravity may be described by formulating a gauge theory of the affine group. However, one has to ensure that the group, i.e. the Lie-algebra valued connection, applies to spacetime – is *soldered* to spacetime. This is done by splitting the connection into a linear part $\Gamma_\alpha{}^\beta$ (with matrix indices ${}_\alpha{}^\beta$ that work on the basis e_α of the local tangent space) and an inhomogeneous part ϑ^α (that replaces the holonomic coframe dx^α and thereby realizes a translational gauge). Analogously, the field strength splits into the curvature $R_\alpha{}^\beta$ and the torsion T^α. Discarding the linear gauge ($\Gamma_\alpha{}^\beta \equiv 0$), the theory reduces to teleparallelism.

Again, so far g is merely a name of a dimension introduced here. However, in the case of electromagnetism, it may be replaced by the *unit* e. Now it is easy to display the dimensions of the components of $A \equiv A^a\,\lambda_a \equiv A_i{}^a\,\lambda_a\,dx^i$ and $F \equiv F^a\,\lambda_a \equiv F_{ij}{}^a\,\lambda_a\,dx^i \wedge dx^j$. You will find them in table 2.

In Yang-Mills theories a lagrangian typically describes propagating gauge fields, i.e. it is proportional to a square term of F. For generality we only assume $\mathcal{L} = \langle F \wedge H \rangle = F^a \wedge H_a$, where we introduced the excitation H, which is a $\mathcal{G}$-valued 2-form, and the metric $\langle\ ,\ \rangle$ in $\mathcal{G}$. We read off the dimension of the excitation $H \equiv H^a\,\lambda_a \equiv H_{ij}{}^a\,\lambda_a\,dx^i \wedge dx^j$ and of the Noether current $\Sigma_a := \delta\mathcal{L}/\delta A^a$. For consistency, the dimension of the metric has to be $[\langle\ ,\ \rangle] = \hbar^2/g^2$. It follows $[\langle \lambda_a, \lambda_b \rangle] = 1$. The dimension of $[H]/[F] = g^2/\hbar$ may be interpreted as the dimension of the coupling constant $1/\kappa$ of a dynamical lagrangian with $H \approx 1/\kappa\ {}^\star F$.

In the case of electrodynamics, we have only one index $a = 0$ and we set $[\lambda_0] = e/\hbar$. We see that the *components* F^a, H^a, and Σ_a have the conventional dimensions, whereas the dimensions of the fields F, H, Σ are more unfamiliar. In the case of a translational gauge theory, we assign the dimension $1/l$ to the generators and find that $[1/\kappa] = \hbar/l^2$. Since this dimensionality includes a length dimension, perturbation theory does not work.

When embedding electrodynamics in an extra dimension à la Kaluza-Klein, the $U(1)$ gauge is directly represented by the translation along the 5th dimension. We can introduce a length *unit* l_5 of this 5th dimension by identifying $e/\hbar = 1/l_5$. This is a geometrical interpretation of the electric unit e as *phase*/2π *per length*. Besides, if the 5th dimension is $U(1)$ with perimeter L_5, it seems natural that this 'phase/2π per length'-unit e is quantized in quanta of $1/L_5$. Hence $l_5 = \hbar L_5$.

	in general	in electrodynamics	in translational gauge theories
$[\mathcal{L} := F^a \wedge H_a] \quad =:$	$\hbar$	Wb C	$\hbar$
$[\lambda_a] \quad =:$	$g/\hbar$	1/Wb	$1/l$
$[A = A^a \lambda_a] \quad \equiv$	1	1	1
$[F = F^a \lambda_a]$	1	1	1
$[H = H^a \lambda_a]$	$g^2/\hbar$	C/Wb	$\hbar/l^2$
$[A^a = A_i{}^a\, dx^i]$	$\hbar/g$	Wb	l
$[F^a = F_{ij}{}^a\, dx^i \wedge dx^j]$	$\hbar/g$	Wb	l
$[H^a = H_{ij}{}^a\, dx^i \wedge dx^j]$	g	C	$\hbar/l$
$[\Sigma_a = \delta\mathcal{L}/\delta A^a]$	g	C	$\hbar/l$
$[\langle\ ,\ \rangle] = 1/[\lambda]^2$	$\hbar^2/g^2$	Wb^2	l^2
$[\langle F, H\rangle]$	$\hbar$	Wb C	$\hbar$
$[1/\kappa] = [H]/[F] = [H^a]/[F^a]$	$g^2/\hbar$	C/Wb	$\hbar/l^2$
$[\mathcal{E}] = [\mathcal{M}] = [F]$	1	1	1
$[\mathcal{E}^a] = [\mathcal{M}^a] = [F^a]$	$\hbar/g$	Wb	l
$[\mathcal{I}]$	g	C	$\hbar/l$

Table 2: The table displays the dimensions of essential fields involved in a gauge theory. In particular, it gives the SI-units in the case of electrodynamics and the dimensions for a translational gauge theory. The first three rows in this table are definitions – the rest is a consequence. The last block includes the dimensions of monopole and topological charges. The SI-units used in electrodynamics are C=Coulomb and Wb=Weber.

Finally we note that the dimension of the hodge star * in n dimensions when applied on a p-form is $[^\star] = l^{n-2p}$.

3 Monopoles, topological and elementary charges

We will define different types of charges, namely monopole, topological, and elementary charges. We identify such charges in internal and external gauge theories and then point out the appropriate analogies.

We start by defining the two types of monopole charges and a topological charge. These charges are properties of the gauge configuration given by the gauge potential A and the

∞ $U(1)$

$U(1) \overset{\text{loc}}{\times} S^2$

slice in spacetime:
the Dirac string

monopole's
location

S^2 $[0,1] \overset{\text{loc}}{\times} S^1$

Figure 1: The field strength of the Dirac monopole [6] $F = p\, d\Omega = p \sin\theta\, d\theta \wedge d\varphi$ has no global potential A with $F = dA$. Dirac concluded that such a monopole must have a *string* (slice in spacetime) attached to it. If we slice spacetime along the negative z-axis, say, F has a regular potential $A = p\,(1-\cos\theta)\, d\varphi$. Alternatively, electomagnetism may be formulated as a gauge theory on a $U(1)$ bundle over spacetime. Topologically the spacetime *around* the (sigular) monopole world path is $(\mathbb{R}^3_{\text{space}} \setminus \{o\}) \times \mathbb{R}_{\text{time}} \sim S^2$, where o denotes the monopole's location. Hence, all field configurations may be classified topologically by investigating $U(1)$ bundles over S^2. It turns out that an integer number (the magnetic charge) classifies all field configurations. The Moebius strip ($[0,1]$ bundle over S^1) allows to visualize a topologically non-trivial bundle.

gauge field strength F.

$$\mathcal{E} := \lim_{r\to\infty} \frac{1}{4\pi} \oint\limits_{S^{n-2}(r)} {}^\star F \,, \qquad \text{quasi-electric monopole charge,} \tag{1}$$

$$\mathcal{M} := \lim_{r\to\infty} \frac{1}{4\pi} \oint\limits_{S^{2}(r)} F \,, \qquad \text{quasi-magnetic monopole charge,} \tag{2}$$

$$\mathcal{C}_I := \lim_{r\to\infty} \frac{1}{4\pi} \oint\limits_{S^{2}(r)\times I} \langle A \wedge F \rangle \,, \qquad \text{Chern-Simons form.} \tag{3}$$

We will discuss I below. The motivation for the definition of $\mathcal{E}$ is obvious from the analogy to Maxwell's inhomogeneous equation. The definition of $\mathcal{M}$ may be motivated by including magnetic charges in Maxwell's theory. Usually this is done by modifying the homogeneous Maxwell equation and introducing a source term on its rhs: $dF = \rho_{\text{mag}}$. But it might be preferable to interpret $\mathcal{M}$ as the topological invariant associated with the first Chern character class $[F]$ in the second cohomology (see below). With this we don't need to introduce magnetic source terms into the homogeneous Maxwell equation but rather interpret magnetic monopoles as a topological feature – which one may visualize as a Dirac string [6] or rather accept as a feature of a $U(1)$-bundle (see figure 1). For an introduction to such mathematical structures one may refer to the textbook [7]. We choose the notion *quasi-electric/-magnetic* to remind us of the analogies with electromagnetism. Since these definitions are general and not restricted to theories of gravitation, we do not choose the notion *gravi*-electric/-magnetic.

The topological charge $\mathcal{C}$ is the fruit of the Chern-Weil theorem which states that it is a topological invariant. We remind the reader of the essential ideas of this theorem. For details see [7]. First, consider the curvature $F \in \Lambda^2(M, \mathcal{G})$ on a principle bundle over the base manifold M and formulate polynomials $P(F)$ of this curvature. Then, search for such polynomials that are invariant under the adjoint action of the structure group G, i.e. $\forall g \in G: \ P(\mathrm{Ad}_g F) = P(F)$. Given such an invariant polynomial of r-th order, the Chern-Weil theorem states the following:

(i) $P(F)$ is closed, i.e. $dP(F) = 0$. Hence we found an element of the $2r$-th cohomology group $[P(F)] \in H^{2r}(M)$. Here, $[P(F)]$ denotes the equivalence class of all $2r$-forms that differ from $P(F)$ only by an exact form. $[P(F)]$ is called *characteristic class*. Note that each monomial in this polynomial is also invariant.
(ii) If we have two curvatures F and F' on the same bundle it follows that $[P(F)] = [P(F')]$. This means that the characteristic class $[P(F)]$ is independent of F and depends only on the topology of the bundle. It is a topological invariant.
(iii) Since $P(F)$ is closed we find a *local* potential on the subset U of M: $P(F) = dQ\big|_U$. It follows that $[Q]$ is an element of the $(2r{-}1)$-th cohomology $H^{2r-1}(\partial U)$ and is thus a topological invariant of ∂U. Q is called *Chern-Simons form.*

In fact, we find the invariant polynomials (or monomials) $P_1(F) = F$ and $P_2(F) = \mathrm{tr}(F \wedge F)$, the first of which is called *1st Chern character class* and the second *1st Pontrjagin class.* We also find the Chern-Simons form $\mathrm{tr}(A \wedge F)$ of the 1st Pontrjagin class.

Hence, the 1st Chern character class $[F]$ is an element of the 2nd cohomology. The intergration of F over a closed 2-plane S^2, i.e. the quasi-magnetic monopole charge $\mathcal{M}$, thus leads to a number that specifies the cohomology class. Similarly, the Chern-Simons form $\langle A \wedge F \rangle$ is an element of the 3rd cohomology and we need a closed 3-plane for integration. In the case of a singular monopole world path in a $U(1)$ bundle, a natural choice for this 3-plane is $S^2 \times U(1)$, with $|U(1)| = L_5$. The integration of the Chern-Simons term $\mathcal{C}_{U(1)}$ over this plane thus leads to a finite number classifying the cohomology class. Analogously we have a second choice $I = \mathbb{R}_{\mathrm{time}}$ to form a 3-plane $S^2 \times I$. However, this plane is not compact and will not lead to a finite number. We solve this problem by restricting I to a finite time interval I_T with $|I_T| = T$. Still, the 3-plane $S^2 \times I_T$ is not closed and, strictly speaking, C_{I_T} may not be considered a topological invariant. Thus we have to act with some caution.

Let us discuss the notion of an elementary charge. One of the most beautiful things in physics is the success of particle physics in classifying all particles with the help of representation theory. This algebraic approach simply postulates that objects in nature must be an element of a representation of some symmetry. Objects (particles/states) that are inseparable are called *elementary.* This notion turns out to coincide with the mathematical notion of *irreducibility.* Both mean inseparable without loosing the symmetry.

With elementary charge we denote those invariants that classify an elementary particle, i.e. the irreducible representation the particle is an element of. Such a classification can be performed by finding all Casimir operators in the group algebra. These are polynomials of the group generators and commute with every group element. Hence, their eigenvalues, when applied on some particle field, are invariant under all symmetry transformations.

The Poincaré group, for example, has the Casimir operators

$$C_1 := P_\mu P^\mu \ , \tag{4}$$

$$C_2 := W_\mu W^\mu \quad \text{with} \quad W_\mu := \frac{1}{2}\epsilon_{\mu\alpha\beta\gamma} L^{\alpha\beta} P^\gamma \ . \tag{5}$$

Here the translation operator P^μ represents the particle momentum, $L^{\alpha\beta}$ are the generators of Lorentz rotations, and the so-called Pauli-Lubanski vector W^μ represents the particle spin. If nature incorporates the Poincaré symmetry, all particles can be classified by eigenvalues of C_1 (mass square) and C_2 (spin square). The classification with respect to their mass is guaranteed by the Dirac equation (for the Dirac spinor representation) or the Klein-Gordon equation (for the scalar representation). All these equations require the dimension $\hbar/l$ for the mass parameter. (We take $c = 1$.)

In general, if C is a polynomial of rth order of the group generators and if $\mathcal{I}^r$ is an invariant eigenvalue of C, i.e. $(\hbar^r C - \mathcal{I}^r)\,\phi = 0$ for some eigenvector ϕ, then we call $\mathcal{I}$ an elementary charge. If we assume that C is built from generators with dimension $[\lambda]$, the dimension of $\mathcal{I}$ is $[\mathcal{I}] = \hbar\,[\lambda]$. This leads to the remarkable relation between the dimension of an elementary charge and that of a monopole charge:

$$[\mathcal{I}] = [1/\kappa]\,[\mathcal{E}] \ . \tag{6}$$

Some further comments on mass as an elementary charge and its dimension: First, the dimension of the mass parameter $[m] = \hbar/l$ may be called *phase/*2π *per length.* In fact, the most obvious argument for this interpretation is the point particle action $\int m\,ds$. In this picture, if you identify a world path with a strap, then mass is the twist of this strap per length. Also note that $\lambda_c = \hbar/m$ may be identified with the Compton *wave* length of the particle. Second, in 5D Kaluza-Klein space the electric charge q is just as well an eigenvalue of the Casimir operator of the translation along the 5th dimension. In this view, electric charge is very similar to mass. Just as mass measures the horizontal momentum, the electric charge measures the vertical momentum. In fact, Bleeker [8] defined electric charge as the 'vertical velocity' of a path on a $U(1)$-bundle. Third, Rosen [10] introduced a *massive* Klein-Gordon (i.e. Proca) field by multiplying a *phase* $\exp(-i\,m_0\,t)$ to a real scalar (1-form) field ϕ. Unfortunately, Rosen does not motivate this in a very detailed manner. Very interestingly, in the case of flat spacetime, the mass m_0 introduced by attaching this phase $\exp(-i\, m_0\, t)$ to the scalar field ϕ cancels with the mass m_1 introduced by adding a square term $(m_1)^2(\phi \wedge {}^\star\phi)$ to the lagrangian.

4 Monopole charges of the translational gauge

We can now apply the charge definitions to analyze standard solutions of Einstein gravity for monopole charges. We concentrate on a subclass of the Plebanski-Demianski class of solutions including the Kerr-Newman and Taub-NUT solutions. For the monopole analysis we formulate them as a solution of a translational gauge theory of gravity, namely teleparallelism, and find, indeed, quasi-electric and quasi-magnetic monopoles in the gauge of some translations.

In spherical coordinates (t, r, θ, φ) the Kerr-Newman metric with mass parameter m, Kerr parameter j, electric charge q, and magnetic charge p reads

$$g = \vartheta^{\hat 0} \otimes \vartheta^{\hat 0} - \vartheta^{\hat 1} \otimes \vartheta^{\hat 1} - \vartheta^{\hat 2} \otimes \vartheta^{\hat 2} - \vartheta^{\hat 3} \otimes \vartheta^{\hat 3}\ , \tag{7}$$

$$\vartheta^{\hat 0} = \frac{\mathcal{Q}}{\Delta}\, d\tau\ , \quad \vartheta^{\hat 1} = \frac{\Delta}{\mathcal{Q}}\, dr\ , \quad \vartheta^{\hat 2} = \frac{\Delta}{\mathcal{P}}\, \sin\theta\, d\theta\ , \quad \vartheta^{\hat 3} = \frac{\mathcal{P}}{\Delta}\, d\sigma\ , \tag{8}$$

$$d\tau = dt - j\sin^2\theta\, d\varphi\ , \quad d\sigma = (r^2 + j^2)\, d\varphi - j\, dt\ , \tag{9}$$

$$\mathcal{Q}^2 = r^2 - 2mr + j^2 + \frac{1}{4}\,(q^2 + p^2)\ , \quad \mathcal{P} = \sin\theta\ , \quad \Delta^2 = r^2 + j^2\cos^2\theta\ . \tag{10}$$

This notation might confuse at first. It is the direct analog of the notation Plebanski and Demianski used in their paper [9]. It has a clear structure and can easily be modified into other solutions of the Plebanski-Demianski class. The metric solves the coupled Einstein-Maxwell equations if we choose the electromagnetic potential as

$$A = \frac{1}{\Delta^2}\big(q\, r\, d\tau + p\cos\theta\, d\sigma\big)\ . \tag{11}$$

This potential is the analog of the potential $A = q/r\, dt + p\cos\theta\, d\varphi$ of an electric and magnetic charge in flat spacetime.

For the monopole analysis, we translate this solution into a 5D Kaluza-Klein-type teleparallelism. This simply means that we add a 5th dimension that represents the electromagnetic part of the theory:

$$g = \vartheta^{\hat 0} \otimes \vartheta^{\hat 0} - \vartheta^{\hat 1} \otimes \vartheta^{\hat 1} - \vartheta^{\hat 2} \otimes \vartheta^{\hat 2} - \vartheta^{\hat 3} \otimes \vartheta^{\hat 3} - \vartheta^{\hat 5} \otimes \vartheta^{\hat 5}\ , \tag{12}$$

$$\vartheta^{\hat 5} = dx^5 + \frac{1}{\Delta^2}\big(q\, r\, d\tau + p\cos\theta\, d\sigma\big)\ . \tag{13}$$

The 5th covector ϑ^5 represents the electromagnetic gauge potential. The field strength of this gauge theory is the torsion $T^\alpha = d\vartheta^a$. The configuration solves the vacuum field equation $dH^a = 0$ of the teleparallelism theory. Here, H^a is the excitation of the translational gauge and is composed out of the three irreducible pieces of T^a such that the theory is equivalent to 5D Einstein gravity:

$$H^a = \frac{1}{\kappa}{}^\star\left({}^{(1)}T^a - 3\,{}^{(2)}T^a + \frac{5}{2}\,{}^{(3)}T^a\right) \quad \text{or} \quad H_\alpha = -\frac{1}{2}K^{\mu\nu} \wedge \eta_{\alpha\mu\nu}\ , \tag{14}$$

where $K^{\mu\nu}$ is the contortion. For details see [4] or [5].

The following charges for this gauge configuration are calculated straightforwardly:

$$\mathcal{E} = -m\,\partial_t\ - j\frac{\pi}{4}\partial_\varphi\ + q\,\partial_5\ , \quad \mathcal{M} = -p\,\partial_5\ , \quad \mathcal{C}_{U(1)} = -p\,L_5\ , \quad \mathcal{C}_{I_T} = 0\ . \tag{15}$$

Consider $\mathcal{E}$ and recognize that we have a quasi-electric monopole charge $\mathcal{E}^t = -m$ in the time translation, a quasi-electric monopole charge $\mathcal{E}^\varphi = -j\frac{\pi}{4}$ in the translation along ∂_φ (which is actually a rotation and the charge represents an angular momentum)[1], and a (quasi-)electric monopole charge $\mathcal{E}^5 = q$ in the translation along ∂_5 (i.e. the $U(1)$ gauge of electrodynamics). In this solution all Killing vectors carry quasi-electric monopole charges. In fact, it seem quite plausible that the elementary charges of the three Casimir operators (momentum square, Pauli-Lubanski square, and the 5th translation) of 5D Kaluza-Klein/Poincaré symmetry are the sources of the quasi-electric monopole charges of the Killing vectors that correspond to these Casimirs.

As we are interested in dimensions, we find that the mass parameter has dimension $[m] = l$, the angular momentum per mass unit has dimension $[j] = 1$, and, if we measure the length along the 5th dimension in units of l_5, the electric charge has dimension $[q] = l_5$. In the previous dimensional discussion of electrodynamics, we defined $1/l_5 = e/\hbar$ and $[1/\kappa] = e^2/\hbar = \hbar/l_5^2$. Here our results are consistent with eq. (6): The dimension of the elementary charge $[\mathcal{I}] = e = \hbar/l_5$ is equal to the coupling constant $[1/\kappa]$ times the dimension of the quasi-electric monopole charge $[\mathcal{E}] = [q] = l_5$. The same holds for the mass.

Considering $\mathcal{M}$ we are not surprised that $\mathcal{M}^5 = -p$ is a (quasi-)magnetic monopole charge of the 5th translation. The non-trivial Chern-Simons form $\mathcal{C}_{U(1)}$ confirms the topological feature of magnetic monopoles in the $U(1)$-bundle.

Let us turn to the Taub-NUT solution with mass parameter m, NUT parameter n, and electric charge q. Within the previous notation, i.e. with the coframe and metric defined

[1] Usually, one associates a *gravi-magnetic* or *gravito-magnetic* effect with the gravitational field of the Kerr solution. This is sensible since the rotating mass produces a field that is in analogy to the magnetic field produced by rotating electrons. However, rotating mass is not an analogy to a magnetic *monopole.* Instead, our calculation definitely proves that it is rather an analogy to an electric monopole – but with respect to the gauge of *orbital translations.*

electric monopole	Schwarzschild solution
$A = -\frac{q}{r}\,dt$ $F = -\frac{q}{r^2}\,dt \wedge dr$	$\Gamma^{(T)\hat{0}} = \vartheta^{\hat{0}} - dt = \left(\sqrt{1-\frac{2m}{r}} - 1\right) dt \underset{r\to\infty}{\longrightarrow} -\frac{m}{r}\,dt$ $T^t = -\frac{m}{r^2}\,dt \wedge dr$
magnetic monopole	Taub-NUT solution
$A = p(1-\cos\theta)\,d\varphi$ $F = dA = p\,d\Omega$	$\Gamma^{(T)\hat{0}} = \vartheta^{\hat{0}} - dt \underset{r\to\infty}{\longrightarrow} 2n(1-\cos\theta)\,d\varphi$ $T^t \underset{r\to\infty}{\longrightarrow} 2n\,d\Omega$

Table 3: The table compares the electric monopole with the Schwarzschild solution and the magnetic monopole with the Taub-NUT solution. The gravitational solutions are presented in a teleparallel formalism. The analogies between the electro-magnetic field strength F and the field strength of time translation $T^{\hat{0}}$ confirm our interpretation of the mass parameter m and the NUT parameter n. The identification of $\vartheta^{\hat{0}} - dt$ with the gauge potential of time translation takes the soldering into account.

in (12, 8, 13), the solution reads

$$d\tau = dt - 2n\cos\theta\,d\varphi\,, \quad d\sigma = (r^2+n^2)\,d\varphi\,, \tag{16}$$

$$p = 0\,, \quad \mathcal{Q}^2 = r^2 - 2mr - n^2 + q^2/4\,, \quad \mathcal{P} = \sin\theta\,, \quad \Delta^2 = r^2 + n^2\,. \tag{17}$$

The result of the monopole analysis is

$$\mathcal{E} = -m\,\partial_t\; + q\,\partial_5\,, \quad \mathcal{M} = -2n\,\partial_t\,, \quad \mathcal{C}_{U(1)} = 0\,, \quad \mathcal{C}_{I_T} = -2n\,T\,. \tag{18}$$

This clearly presents the NUT parameter n as a quasi-magnetic monopole charge of the time translation. Table 3 gives another illustration of these results.

The reader may verify and have an insight into the calculations by investigating our input files for the computer algebra system Reduce. For this, download the files `kerrnut.exi` and `magtools` from the internet page
`http://www.thp.uni-koeln.de/~mt/work/1999charge/` .
We use Reduce version 3.6 together with the Excalc package. You find help pages at
`http://www.uni-koeln.de/REDUCE/3.6/doc/reduce/` and
`http://www.uni-koeln.de/REDUCE/3.6/doc/excalc/` .
Contact `mt@thp.uni-koeln.de` for any problems in this context.

5 Discussion

The main results of this paper are summarized in table 2 and eqs. (6), (15), and (18). Also the observation concerning the correspondence of elementary and quasi-electric monopole charges on the one hand, and of Casimir operators and Killing vectors on the other hand is important. I want to add some comments:

We proved that in the Plebanski-Demianski class of solutions [9] (when reformulated as teleparallel solutions) the five parameters m, n, q, p, and j may be related to monopole charges. Unfortunately, we could not confirm the same for the acceleration parameter a. (The reason might be the topologically non-trivial coordinate transformation eq. (4.4) in [9].) However, for consistency we may expect that a relates to a quasi-magnetic charge of the 'orbital translation' along ∂_φ. Assuming this, we agree with Plebanski and Demianski on their ordering of the parameters: The six parameters should be ordered as three pairs $(m,\, n)$, $(j,\, a)$, and $(q,\, p)$ each pair of which belongs to the time translation, the orbital translation, and the $U(1)$-translation, respectively. In each pair the first parameter denotes the quasi-electric charge and the second parameter the quasi-magnetic charge of these translations.

It seems to be commonly accepted that mass represents a quasi-electric charge. Also in the Riemannian formulation of Einstein gravity there are arguments in favour of this interpretation. However, I believe that the Riemannian formulation can impossibly recover mass as a quasi-electric charge of the *time-translation*. Hence, the teleparallel formulation of Einstein gravity has considerable advantages over the Riemannian formulation. Also the *explicit* presentation of the NUT parameter as quasi-magnetic monopole charge of the time-translation seems new.

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